OCR
RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level
former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS 5506

Pure Mathematics 6

Thursday 14 JUNE 2001 Morning 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

This question paper consists of 3 printed pages and 1 blank page.

HN

[Turn over

Option 1: Limiting Processes

1 (a) Use L'Hôpital's rule to find $\lim_{x \to e} \frac{\ln x - 1}{x^3 - e^3}$. [4]

(b) You are given that $xe^{-x} \to 0$ as $x \to \infty$.

By considering $\left(xe^{-x}\right)^3$, show that $y^3 e^{-y} \to 0$ as $y \to \infty$. [4]

(c) (i) Explain in detail how $\sum_{r=1}^{n} \frac{r^2}{2n^3 - r^3}$ is related to an area under the curve $y = \frac{x^2}{2 - x^3}$. [5]

(ii) Show that, as $n \to \infty$, $\sum_{r=1}^{n} \frac{r^2}{2n^3 - r^3}$ tends to a limit L, and evaluate this limit. [3]

(iii) Show that $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^5}{2n^6 - r^6} = \frac{1}{2}L$. [4]

Option 2: Multi-Variable Calculus

2 A surface has equation $z = x^3 + 3x^2 + 6xy + y^2$.

(i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. [2]

(ii) Find the coordinates of the two stationary points on the surface. [6]

(iii) Find the equation of the normal line at the point P(–1, 1, –3). [4]

(iv) Find the other point Q on the surface where the normal line is parallel to the normal line at P, and find the equation of the tangent plane at Q. [8]

Option 3: Vectors and Matrices

3 Three points A, B and C have coordinates (9, 1, 5), (5, –2, 3) and (–3, 10, 8) respectively. The point $P(9+\lambda, 1+3\lambda, 5-4\lambda)$, where $\lambda > 0$, is a general point on a straight line starting at A.

(i) Find the volume of the tetrahedron ABCP. State whether $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AP}$ (in that order) is a right-handed or left-handed set of vectors. [6]

(ii) Find the shortest distance from P to the plane ABC. [4]

(iii) Find, in terms of λ, the shortest distance between the lines CP and AB. Show that, when λ is small, this distance is approximately 5λ; and show that, when λ is large, this distance is approximately 15. [10]

Option 4: Differential Geometry

4 **(a)** A curve has intrinsic equation $s = 5\sin\psi$, where s is the arc length measured from the origin, and ψ is the angle between the tangent and the x-axis.

Show that the radius of curvature at the point (s, ψ) on the curve is $\sqrt{25 - s^2}$. [5]

(b) **(i)** Find the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a general point $(a\cos\theta, b\sin\theta)$. [4]

(ii) Find parametric equations for the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [6]

(iii) Hence or otherwise show that the centre of curvature corresponding to the point (8,3) on the ellipse $\frac{x^2}{100} + \frac{y^2}{25} = 1$ is $\left(\frac{96}{25}, -\frac{81}{25}\right)$ and find the radius of curvature at the point (8,3). [5]

Option 5: Abstract Algebra

5 The real vector space V consists of all quadratic polynomials

$$f(x) = ax^2 + bx + c \quad \text{such that} \quad f(2) = 0.$$

Addition of polynomials, and multiplication of a polynomial by a scalar, are defined in the usual way.

Three elements of V are f_1, f_2, f_3, where

$$f_1(x) = x - 2, \quad f_2(x) = x^2 - 4, \quad f_3(x) = 2x^2 - 7x + 6.$$

(i) Show that $\{f_1, f_2, f_3\}$ is a linearly dependent set. [3]

(ii) Show that $\{f_1, f_2\}$ is a basis for V. [4]

The real vector space W consists of all two dimensional vectors $\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$, and the mapping $T : V \to W$ is defined by $T(f) = \begin{pmatrix}f(3) \\ f(5)\end{pmatrix}$.

(iii) Show that T is a linear mapping. [4]

(iv) Find the matrix **M** associated with T and the bases

$$\{f_1, f_2\} \text{ of } V \quad \text{and} \quad \left\{\begin{pmatrix}1 \\ 0\end{pmatrix}, \begin{pmatrix}0 \\ 1\end{pmatrix}\right\} \text{ of } W.$$ [4]

(v) By considering $\mathbf{M}^{-1}$, find the quadratic polynomial $f(x) = ax^2 + bx + c$ which satisfies $f(2) = 0$, $f(3) = p$, $f(5) = q$, expressing the coefficients a, b and c in terms of p and q. [5]

Mark Scheme

1 (a)	Limit is $\lim_{x\to e}\dfrac{1/x}{3x^2}$ $=\dfrac{1/e}{3e^2}$ $=\dfrac{1}{3e^3}$	M1 A1 M1 A1 cao 4	Differentiating num and denom For $1/x$ and $3x^2$ Considering $\dfrac{f'(e)}{g'(e)}$
(b)	$(xe^{-x})^3 \to 0 \quad (\text{as } x\to\infty)$ $(xe^{-x})^3 = x^3e^{-3x} = \frac{1}{27}y^3e^{-y}$ where $y=3x$ Hence $y^3e^{-y}\to 0$ as $y\to\infty$	B1 M1A1 A1 4	 For completion
(c)(i)	$\dfrac{r^2}{2n^3-r^3}=\dfrac{1}{n}\dfrac{(r/n)^2}{2-(r/n)^3}$ Sum is the area of rectangles of width $1/n$ above the curve between $x=0$ and $x=1$	B2 B1 B1 B1 5	
(ii)	L is area under curve between $x=0$ and $x=1$ $L=\int_0^1\dfrac{x^2}{2-x^3}\,dx=\left[-\frac{1}{3}\ln(2-x^3)\right]_0^1$ $=\frac{1}{3}\ln 2$	B1 M1 A1 cao 3	*May be implied* For integral is $k\ln(2-x^3)$
(iii)	$\dfrac{r^5}{2n^6-r^6}=\dfrac{1}{n}\dfrac{(r/n)^5}{2-(r/n)^6}$ So $\lim_{n\to\infty}\sum_{r=1}^{n}\dfrac{r^5}{2n^6-r^6}$ is the area under $y=\dfrac{x^5}{2-x^6}$ between $x=0$ and $x=1$ Limit is $\int_0^1\dfrac{x^5}{2-x^6}\,dx=\left[-\frac{1}{6}\ln(2-x^6)\right]_0^1$ $=\frac{1}{6}\ln 2=\frac{1}{2}L$	M1 A1 M1 A1 cao 4	*Working essential* For integral is $k\ln(2-x^6)$

2 (i)	$\dfrac{\partial z}{\partial x} = 3x^2 + 6x + 6y\,,\ \dfrac{\partial z}{\partial y} = 6x + 2y$	B1B1 **2**	
(ii)	$3x^2 + 6x + 6y = 0$ and $6x + 2y = 0$ So $y = -3x$ and $3x^2 - 12x = 0$ Stationary pts are $(0\,,0\,,0)$ and $(4\,,-12\,,-32)$	M1 M1A1 ft A3 cao **6**	Give A1 for 2 coordinates correct Give A2 for 4 coordinates correct
(iii)	At P, $\dfrac{\partial z}{\partial x} = 3\,,\ \dfrac{\partial z}{\partial y} = -4$ Normal line is $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$	B1 cao M1A1 ft A1 ft **4**	For direction
(iv)	$3x^2 + 6x + 6y = 3\,,\ 6x + 2y = -4$ So $y = -3x - 2$ and $3x^2 - 12x - 15 = 0$ $x = -1\,,\ 5$ Q is $(5\,,-17\,,-21)$ Tangent plane at Q is $3x - 4y - z = 15 + 68 + 21$ $3x - 4y - z = 104$	M1A1 ft M1 M1 A1 cao M1 M1 A1 cao **8**	Obtaining equation in x (or y) Solving to obtain x (or y) For $3x - 4y - z$ Finding constant term

3 (i)	Volume is $\frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AP}$	M1	Vector product of two sides
	$=\frac{1}{6}\left[\begin{pmatrix}-4\\-3\\-2\end{pmatrix}\times\begin{pmatrix}-12\\9\\3\end{pmatrix}\right].\begin{pmatrix}\lambda\\3\lambda\\-4\lambda\end{pmatrix}=\frac{1}{6}\begin{pmatrix}9\\36\\-72\end{pmatrix}.\begin{pmatrix}\lambda\\3\lambda\\-4\lambda\end{pmatrix}$	A1	Vector product correct
		M1	Scalar triple product
	$=\frac{1}{6}\times 405\lambda$	M1	Fully correct method (including $\frac{1}{6}$)
	$=\dfrac{135\lambda}{2}$	A1 cao	
	Since $(\overrightarrow{AB}\times\overrightarrow{AC}).\overrightarrow{AP}>0$, it is a right-handed set	B1 **6**	No explanation required
(ii)	Normal vector is $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}9\\36\\-72\end{pmatrix}=9\begin{pmatrix}1\\4\\-8\end{pmatrix}$	B1	
	Unit normal vector is $\hat{\mathbf{n}}=\frac{1}{9}\begin{pmatrix}1\\4\\-8\end{pmatrix}$		
	Distance is $\overrightarrow{AP}.\hat{\mathbf{n}}=\begin{pmatrix}\lambda\\3\lambda\\-4\lambda\end{pmatrix}.\frac{1}{9}\begin{pmatrix}1\\4\\-8\end{pmatrix}$	M1A1 ft	Any completely correct method, e.g. $\dfrac{(9+\lambda)+4(1+3\lambda)-8(5-4\lambda)+27}{\sqrt{1^2+4^2+8^2}}$
	$=5\lambda$	A1 cao **4**	
(iii)	$\overrightarrow{CP}\times\overrightarrow{AB}=\begin{pmatrix}12+\lambda\\-9+3\lambda\\-3-4\lambda\end{pmatrix}\times\begin{pmatrix}-4\\-3\\-2\end{pmatrix}=\begin{pmatrix}9-18\lambda\\36+18\lambda\\-72+9\lambda\end{pmatrix}$	M1A1	
	$=9\begin{pmatrix}1-2\lambda\\4+2\lambda\\-8+\lambda\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}-12\\9\\3\end{pmatrix}$		
	Distance is $\dfrac{\begin{pmatrix}-12\\9\\3\end{pmatrix}.\begin{pmatrix}1-2\lambda\\4+2\lambda\\-8+\lambda\end{pmatrix}}{\sqrt{(1-2\lambda)^2+(4+2\lambda)^2+(-8+\lambda)^2}}$	M1A1 ft M2	For magnitude Fully correct method
	$=\dfrac{45\lambda}{\sqrt{9\lambda^2-4\lambda+81}}$	A1 cao	
	When λ is small, this is approx $\dfrac{45\lambda}{\sqrt{81}}=5\lambda$	M1 A1 (ag)	Approximating (either case)
	When λ is large, this is approx $\dfrac{45\lambda}{\sqrt{9\lambda^2}}=15$	A1 (ag) **10**	

4 (a)	$\rho = \dfrac{ds}{d\psi}$ $= 5\cos\psi$ $= 5\sqrt{1-\sin^2\psi}$ $= 5\sqrt{1-\left(\dfrac{s}{5}\right)^2} = \sqrt{25-s^2}$	M2 A1 M1 A1 (ag) 5	
(b)(i)	$\dfrac{dy}{dx} = \dfrac{b\cos\theta}{-a\sin\theta}$ Normal is $y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$ $by = (a\tan\theta)x - (a^2-b^2)\sin\theta$	M1 A1 M1 A1 4	Method for finding gradient Obtaining equation of normal
(ii)	Differentiating partially with respect to θ, $0 = (a\sec^2\theta)x - (a^2-b^2)\cos\theta$ $x = \left(\dfrac{a^2-b^2}{a}\right)\cos^3\theta$ $y = \dfrac{a}{b}\tan\theta\left(\dfrac{a^2-b^2}{a}\right)\cos^3\theta - \left(\dfrac{a^2-b^2}{b}\right)\sin\theta$ $= \left(\dfrac{a^2-b^2}{b}\right)\sin\theta(\cos^2\theta - 1)$ $= -\left(\dfrac{a^2-b^2}{b}\right)\sin^3\theta$	M1 A1 M1 A1 M1 A1 6	Obtaining x in terms of θ Obtaining y in terms of θ Give A1 for x, y both correct but unsimplified
	OR $\rho = \dfrac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}} = \dfrac{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}}{ab}$ M1A1 $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}\begin{pmatrix}-b\cos\theta \\ -a\sin\theta\end{pmatrix}$ M1 $\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}a\cos\theta \\ b\sin\theta\end{pmatrix} + \dfrac{a^2\sin^2\theta + b^2\cos^2\theta}{ab}\begin{pmatrix}-b\cos\theta \\ -a\sin\theta\end{pmatrix}$ M1 A1 $x = \left(\dfrac{a^2-b^2}{a}\right)\cos^3\theta, \quad y = -\left(\dfrac{a^2-b^2}{b}\right)\sin^3\theta$ A1		Or finding κ
(iii)	$a = 10,\ b = 5$; at $(8, 3)$ $\cos\theta = \frac{4}{5},\ \sin\theta = \frac{3}{5}$ Centre of curvature has $x = \left(\dfrac{100-25}{10}\right)\left(\dfrac{4}{5}\right)^3, \quad y = -\left(\dfrac{100-25}{5}\right)\left(\dfrac{3}{5}\right)^3$ i.e. $\left(\dfrac{96}{25}, -\dfrac{81}{25}\right)$ $\rho = \sqrt{\left(8-\dfrac{96}{25}\right)^2 + \left(3+\dfrac{81}{25}\right)^2}$ $= \sqrt{\dfrac{35152}{625}} = \dfrac{52\sqrt{13}}{25} \approx 7.5$	M1 A1 ft A1 (ag) M1 A1 5	M1A2 cao from a fresh start M1A1 cao from a fresh start

5 (i)	$f_3(x) = 2x^2 - 7x + 6 = 2(x^2 - 4) - 7(x - 2)$ $f_3 = 2f_2 - 7f_1$ so the set is linearly dependent	M1A1 A1	3	
(ii)	$\lambda_1 f_1 + \lambda_2 f_2 = 0 \Rightarrow \lambda_1(x-2) + \lambda_2(x^2 - 4) = 0$ $\Rightarrow \lambda_2 x^2 + \lambda_1 x + (-2\lambda_1 - 4\lambda_2) = 0$ $\Rightarrow \lambda_1 = \lambda_2 = 0$ so the set is linearly independent If $f \in V$ then $f(x) = ax^2 + bx + c$ and $4a + 2b + c = 0$ so $f(x) = ax^2 + bx - 4a - 2b = a(x^2 - 4) + b(x - 2)$ $= af_1 + bf_2$ so the set spans V Hence it is a basis	B1 M1A1 A1	4	
(iii)	$T(f+g) = \begin{pmatrix}(f+g)(3)\\(f+g)(5)\end{pmatrix} = \begin{pmatrix}f(3)+g(3)\\f(5)+g(5)\end{pmatrix}$ $= \begin{pmatrix}f(3)\\f(5)\end{pmatrix} + \begin{pmatrix}g(3)\\g(5)\end{pmatrix} = T(f) + T(g)$ $T(\lambda f) = \begin{pmatrix}(\lambda f)(3)\\(\lambda f)(5)\end{pmatrix} = \begin{pmatrix}\lambda f(3)\\\lambda f(5)\end{pmatrix} = \lambda\begin{pmatrix}f(3)\\f(5)\end{pmatrix} = \lambda T(f)$	M1 A1 M1A1	4	
(iv)	$T(f_1) = \begin{pmatrix}f_1(3)\\f_1(5)\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$, $T(f_2) = \begin{pmatrix}f_2(3)\\f_2(5)\end{pmatrix} = \begin{pmatrix}5\\21\end{pmatrix}$ $\mathbf{M} = \begin{pmatrix}1 & 5\\3 & 21\end{pmatrix}$	M1A1 M1A1	4	
(v)	$\mathbf{M}^{-1}\begin{pmatrix}p\\q\end{pmatrix} = \frac{1}{6}\begin{pmatrix}21 & -5\\-3 & 1\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix} = \frac{1}{6}\begin{pmatrix}21p - 5q\\-3p + q\end{pmatrix}$ We require $T(f) = \begin{pmatrix}p\\q\end{pmatrix}$, so $f(x) = \frac{1}{6}(21p - 5q)f_1(x) + \frac{1}{6}(-3p + q)f_2(x)$ $= \frac{1}{6}(21p - 5q)(x - 2) + \frac{1}{6}(-3p + q)(x^2 - 4)$ $= \frac{1}{6}(-3p + q)x^2 + \frac{1}{6}(21p - 5q)x + (-5p + q)$	M1A1 M1 A1 ft A1 cao	5	

Examiner’s Report

Pure Mathematics 6 (5506)

General Comments

Most candidates performed well on this paper, with about 15% scoring 50 marks or more (out of 60) and only about 25% scoring less than 30 marks. Few candidates appeared to have difficulty in completing the paper in the time allowed. The most popular choice, by far, was questions 2, 3 and 4.

Comments on Individual Questions

Question 1 (Limiting Processes)

This was the least popular question, attempted by about 20% of candidates. It was not well answered, with half of the attempts scoring 8 marks or less (out of 20). L'hôpital's rule was usually applied correctly in part (a), but in part (b) very few candidates used the substitution $y = 3x$ to give a convincing proof. In part (c), several candidates proceeded confidently and scored most of the marks, but very many produced work of little value.

(a) $\frac{1}{3e^3}$; (c)(ii) $\frac{1}{3}\ln 2$.

Question 2 (Multi-Variable Calculus)

This question was extremely well answered, with half the attempts scoring 19 or 20 marks. The only common error, in part (iv), was to try to find Q as the point where the normal line at P meets the surface again.

(i) $\frac{\partial z}{\partial x} = 3x^2 + 6x + 6y$, $\frac{\partial z}{\partial y} = 6x + 2y$; (ii) $(0, 0, 0)$ and $(4, -12, -32)$;

(iii) $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}$; (iv) $(5, -17, -21)$, $3x - 4y - z = 104$.

Question 3 (Vectors)

The techniques required in this question were well known, except that in part (ii) the formula for the distance of a point from a plane was sometimes confused with the formula for the distance of a point from a line. However, most candidates made errors, usually when evaluating vector products, and the average mark for the question was 13. Many candidates showed that they knew how to obtain the approximations in the final part.

(i) $\frac{135\lambda}{2}$, right-handed; (ii) 5λ; (iii) $\frac{45\lambda}{\sqrt{9\lambda^2 - 4\lambda + 81}}$.

Question 4 (Differential Geometry)

The average mark for this question was 10.5. Part (a) was often answered correctly, but some candidates attempted to apply the cartesian formulae to the intrinsic equation. The normal in part (b)(i) was usually correct, although many errors, such as giving the equation of the tangent instead, were made. In part (b)(ii), the majority applied partial differentiation to the equation of the normal, and others found the radius of curvature at a general point and used $\mathbf{c} = \mathbf{r} + \rho\hat{\mathbf{n}}$. Whichever method was used, it was rarely completed without errors. Part (b)(iii) was often omitted; it was surprising how few candidates seemed to realise that the radius of curvature could be found as the distance between the two points given in the question.

(b)(i) $by = (a\tan\theta)x - (a^2 - b^2)\sin\theta$; (ii) $x = \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta, \quad y = -\left(\frac{a^2 - b^2}{b}\right)\sin^3\theta$;

(iii) $\frac{52\sqrt{13}}{25}$.

Question 5 (Vector Spaces)

This question was only attempted by about a quarter of the candidates, and the average mark was 9. There were a few good answers, but the majority answered one or two parts well, usually from parts (i), (iii) and (iv), and scored little in the rest of the question. In part (ii), most candidates claimed to show that the set spanned V by writing $ax^2 + bx + c = a(x^2 - 4) + b(x - 2)$; very few explained why the constant term is correct. Part (v) was usually omitted.

(iv) $\begin{pmatrix} 1 & 5 \\ 3 & 21 \end{pmatrix}$; (v) $a = \frac{1}{6}(-3p + q)$, $b = \frac{1}{6}(21p - 5q)$, $c = -5p + q$.